Instrumentation and VEGAS

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6.5 Upper limit

6.5.1 Physical Reasoning

The flux upper limit is generally reported as

$$F_{>E_{\rm thr}} = \int_{E_{\rm thr}}^{\infty} \frac{dN}{dE} dE$$

The quantities to find for the instrument are the threshold energy and differential flux. The differential flux can take the form of a powerlaw or any other common model. For a powerlaw, we need to compute the normalization using the effective area of the instrument.

$$\begin{aligned} \frac{dN}{dE} &= k_0 \left(\frac{E}{E_0}\right)^{-\gamma} \\ k_0 &= \frac{\langle \text{counts} \rangle}{\langle \text{time} \rangle \langle \text{area} \rangle \langle \text{energy} \rangle} \\ \langle \text{counts} \rangle &= \lambda_{\rm source} \end{aligned}$$

The upper limit calculations (Rolke, Helene, Feldman&Cousins, etc.) use the number of counts On/Off, α and confidence limit (e.g. 95%) to compute the statistical number of counts possible, $\lambda_{\rm lim}$. To estimate the largest possible upper limit, these value should use the whole observed On/Off counts and exposure of the instrument, energy independent. The assumption is that these counts are from the source (not background), which follows the given physical spectral model. To calculate the number of counts from a model:

$$\begin{aligned} \lambda_{\rm source} &= \int_{\rm instrument} \frac{dN}{dE} A(E) dE dt \\ &= t k_0 \int \left(\frac{E}{E_0}\right)^{-\gamma} A(E) dE \end{aligned}$$

Setting $\lambda_{\rm source} = \lambda_{\rm lim}$, we can then find the upper limit normalization to place into the integral flux $F_{>E_{\rm thr}}$ at the top.

$$k_0 = \frac{\lambda_{\rm lim}}{t \int \left(\frac{E}{E_0}\right)^{-\gamma} A(E) dE}$$

Putting it all together,

$$
\begin{aligned}
F_{>E_{\rm thr}} &= \int_{E_{\rm thr}}^{\infty} \frac{dN}{dE} dE \\
&= \int_{E_{\rm thr}}^{\infty} k_0 \left(\frac{E}{E_0}\right)^{-\gamma} dE \\
&= \int_{E_{\rm thr}}^{\infty} \left[\frac{\lambda_{\rm lim}}{t \int \left(\frac{E}{E_0}\right)^{-\gamma} A(E) dE}\right] \left(\frac{E}{E_0}\right)^{-\gamma} dE
\end{aligned}
$$

Since this upper limit is model dependent, the final value could be anything within the model parameter space $\{k_0, \gamma\}$. k_0 is dependent on the count upper limit and γ, so we can vary γ to get a distribution of the range of flux upper limits. This process is approximated by trying 3 γ's to see the spread of flux upper limits, then reporting the energy at which pairs of these fluxes are close to each other (the so-called decorrelation energy). If the energies are exceptionally different ($> 30\%$), a warning is reported to indicate a bias due to the γ dependence. The decorrelation energy is then used in the upper limit integral, which approximates the sum of all the possible fluxes within the model parameter space $\{k_0, \gamma\}$. This is a good estimate for a parameter-independent flux upper limit.

This equation forms the strategy implemented in VEGAS in the next section, where I rename $F_{>E_{\rm thr}}$ to $\Phi_{p,int}$, and $\frac{dN}{dE}$ to $\phi(E)$.

6.5.2 Implementation

Try 3 different spectrum, all power laws with $\gamma_{1,2,3} = [-2.5, -3.5, -4.5]$. EA graph has energies(X) and area points(Y). Split data into hundreds of energy points and interpolate areas between points. Default is 400 points. Go through each energy bin (of hundreds) and get average integral flux for that bin

```
aveSpec1 = fSpec1->Integral()/ (uEdge - lEdge)
```

This is just the normalization of the integral, since none of the physical features are included and energy gets divided out.

$$
\begin{aligned}
\langle S_i \rangle &= \frac{1}{E_u - E_l} \int_{E_l}^{E_u} E^{-\gamma} dE \\
\langle A \rangle_i &= a_i / N_{tot}
\end{aligned}
$$

Sum over all energy bins to get the integral flux (for `aveArea` and energy edges for that bin):

```
fIntegral1 += aveSpec1 * aveArea * (uEdge - lEdge);
```

Repeat for all three power laws and any user-specified. These are the denominators for the photon rates (see product with area and energy.) No data included in this step, just lookup tables for EA and EThresh.

$$\text{fIntegral} = \sum_{E_i} \langle S_i \rangle \langle A \rangle_i * (E_{u,i} - E_{l,i})$$

$$\text{Integral} = \text{fIntegral} * \text{fLiveTime}$$

Get the total livetime and multiply all the "integrals" to get the full denominator of the spectrum K_0. Use ROOT's upper limit packages to get the photon count upper limit with the measured values.

UL Example:

```
TRolke rolke(fConfidenceLevel);
rolke.SetBounding((sDefaultRolkeBounded) ? (true) : (false));
rolke.SetPoissonBkgKnownEff(nON, nOFF, 1 / fAlpha, 1);
fUL = rolke.GetUpperLimit();
```

Convert the photon UL to normalization from the count:

```
// Calculate the normalization factors
fK = fUL / integral;
```

Use this norm in the spectrum and get the integral value.

```
fULint = fK * fSpectrum->Integral(fUserMinEnergy, fUserMaxEnergy);
```

Putting it all together:

$$\begin{aligned}
\phi_{p,int} &= \int_{E_{\text{thr}}}^{\infty} \phi(E) dE \\
&= \int_{E_{\text{thr}}}^{\infty} K_0 (E)^{-\gamma} dE \\
K_0 &= \frac{N_{UL}}{I} \\
I &= T \sum_{i}^{N=400} \langle A \rangle_i (E_{u,i} - E_{l,i}) \langle S_i \rangle \\
&= T * \sum_{i}^{N=400} \langle A \rangle_i (E_{u,i} - E_{l,i}) \frac{1}{E_{u,i} - E_{l,i}} \int_{E_{l,i}}^{E_{u,i}} E^{-\gamma} dE \\
\phi_{p,int} &= \int_{E_{\text{thr}}}^{\infty} \frac{N_{UL}}{T * \left[\sum_i^N \langle A \rangle_i \int_{E_{l,i}}^{E_{u,i}} E'^{-\gamma} dE' \right]} (E)^{-\gamma} dE
\end{aligned}$$

Separately, the decorrelation energy is also calculated:

$$E_{decorr1} = \left(\frac{K_3}{K_1} \right)^{\frac{1}{\gamma_1 - \gamma_3}}$$

$$E_{decorr2} = \left(\frac{K_2}{K_1} \right)^{\frac{1}{\gamma_1 - \gamma_2}}$$

The decorrelation energies are the energies at which each spectrum overlaps: $E_{decorr1}$ for γ_1 for γ_3, and $E_{decorr2}$ for γ_1 for γ_2. If they are more that 30% different, the bias triggers an error. Otherwise the average between $E_{decorr1}$ and $E_{decorr2}$ is reported.

In conclusion, the UL integral uses the same mechanism to compute whether binned or over full range. In binned case, it uses the user-specified index, while over whole range it uses that and the 3 extra indices.

6.5.3 EBL

Applying the deabsorption formula to equation above:

$$\phi_{p,int} = \int_{E_{\rm thr}}^{\infty} \frac{N_{UL}}{T * \left[\sum_i^N \langle A \rangle_i \int_{E_{l,i}}^{E_{u,i}} (E')^{-\gamma} e^{\tau(z,E')} dE'\right]} (E)^{-\gamma} e^{\tau(z,E)} dE$$

In the denominator, this changes the spectrally-averaged effective area at the high energies by a small amount. EBL deabsorption hardens the spectrum as the energy increases, so each fine bin will increase by a small amount. Overall, it is difficult to predict the total change (whether greater or less than original flux) because the actual change in the spectrally-averaged effective area has to compete with the change in flux integral in the numerator.

6.5.4 Time Dependence

Lets assume there is a time dependence on the spectral model $f(t)$, usually a power law with $\beta \simeq -1.3$

$$\begin{aligned} \frac{dN}{dE} &= k_0 \left(\frac{E}{E_0}\right)^{-\gamma} f(t) \\ &= k_0 \left(\frac{E}{E_0}\right)^{-\gamma} t^{\beta} \end{aligned}$$

The estimate source counts is then

$$\begin{aligned} \lambda_{\rm source} &= \int_{\rm instrument} \frac{dN}{dE} A(E) dE dt \\ &= k_0 \int \left(\frac{E}{E_0}\right)^{-\gamma} t^{\beta} A(E) dE dt \\ &= \begin{cases} k_0 \frac{t^{1+\beta}}{1+\beta} \int \left(\frac{E}{E_0}\right)^{-\gamma} A(E) dE & \text{for } \beta \neq -1 \\ k_0 \ln(t) \int \left(\frac{E}{E_0}\right)^{-\gamma} A(E) dE & \text{for } \beta = -1 \end{cases} \end{aligned}$$

Then we fast forward the same analysis to its final form (for now $E_0 = 1\,TeV$ and E in units of TeV, so that it disappears):

$$\phi_{p,int} = \int\int_{E_{\rm thr}}^{\infty} \frac{N_{UL}}{\left[\frac{t'^{1+\beta}}{1+\beta} * \sum_i^N \langle A\rangle_i \int_{E_{l,i}}^{E_{u,i}} E'^{-\gamma} dE'\right]} (E)^{-\gamma} t^{\beta} dE dt$$
$$= \int\int_{E_{\rm thr}}^{\infty} \frac{N_{UL} * (1+\beta)}{\left[\sum_i^N \langle A\rangle_i \int_{E_{l,i}}^{E_{u,i}} E'^{-\gamma} dE'\right]} (E)^{-\gamma} \frac{t^{\beta}}{t'^{1+\beta}} dE dt$$

Assuming $t = t'$, that can be integrated together

$$= \begin{cases} (1+\beta)\ln(t) \int_{E_{\rm thr}}^{\infty} \frac{N_{UL}}{\left[\sum_i^N \langle A\rangle_i \int_{E_{l,i}}^{E_{u,i}} E'^{-\gamma} dE'\right]} (E)^{-\gamma} dE & \text{for } \beta \neq -1 \\ \int_{E_{\rm thr}}^{\infty} \frac{N_{UL}}{\left[\sum_i^N \langle A\rangle_i \int_{E_{l,i}}^{E_{u,i}} E'^{-\gamma} dE'\right]} (E)^{-\gamma} dE & \text{for } \beta = -1 \end{cases}$$

6.5.5 Statistics

In most cases, upper limits are found by calculating the confidence region of a distribution of signal counts. Counts are poissonian, with some rate given by background and signal physics (all sorts of reasons will define the rate itself). The signal counts is found by the difference in counts for different observing modes, On and Off. With this poissonian distribution, likelihood functions for null and hypothesis are created, and then the confidence range is developed with these functions, whether by likelihood ratios or other means.

$$N_{off} = b$$
$$N_{on} = b + s$$
$$N_{excess} = N_{on} - \alpha N_{off}$$
$$\alpha = \frac{A_{on} t_{on}}{A_{off} T_{off}}$$
$$f(N_{on}, N_{off}|s,b) = \frac{(s+b)^{N_{on}}}{N_{on}!} \frac{(b/\alpha)^{N_{off}}}{N_{off}!}$$

The likelihood function is $L(s,b|N_{on},N_{off}) = f(N_{on},N_{off}|s,b)$. This function is maximized over s and b, so the estimators are:

$$(\hat{s},\hat{b}) = (N_{on} - \alpha N_{off}, \alpha N_{off})$$

The profile likelihood function (a.k.a. likelihood ratio)

$$\lambda(s|N_{on},N_{off}) = \frac{L(s,\hat{b}(s)|N_{on},N_{off})}{L(\hat{s},\hat{b}|N_{on},N_{off})}$$

This λ is maximized, and $-2\log\lambda$ is the test statistic (approximate χ^2 distribution via Wilke's theorem.)

Rolke

The Rolke method adds nuisance parameters to the basic set up to allow for statistical uncertainty. Specifically, say we run m events through our Monte Carlo (without background) and find z events surviving. Then we can model the efficiency Z as a binomial random variable and find the complete model to be

$$\begin{aligned} N_{on} &= Poiss(es + b) \\ N_{off} &= Poiss(b/\alpha) \\ Z &= Binomial(m, e) \end{aligned}$$

Then, the modified log-likelihoods $L(s, b, e|N_{on}, N_{off}, Z)$ are maximized over b and e. The log-likelihoods are then used to calculate the profile likelihood function, parametrized by the signal counts. If the s counts are not small, the confidence limit is found by the varying s until the profile likelihood value has changed by the confidence limit. For example, if C.L. = 90%, the profile likelihood will change by 2.706 from the minimum point (at the estimator $\hat{s}$). This technique is unbounded.

If the s counts is negative on the low end, the s counts lower limit is set to 0. In another case, the profile likelihood function has a negative estimator $\hat{s}$, and the upper limit (estimator + C.L. increase in profile likelihood) is also negative. Add 1 count to N_{on} and recompute the profile likelihood function to find the next upper limit (same method as before). Repeat this addition of N_{on} counts until the upper limit is positive. Special care is also needed when $N_{on} = 0$ and/or $N_{off} = 0$. Simply add 1 count to either or both N_{on} and N_{off} and do a linear extrapolation of the profile likelihood. Use this extrapolation to find the upper limit.

If the estimator $\hat{s}$ is negative: 1) unbounded: add the C.L. to the profile likelihood from the position of $\hat{s}$. 2) bounded: add the C.L. to the profile likelihood from $s = 0$. See figure 6.1 for the profile likelihood example with a negative $\hat{s}$.

Feldman & Cousins

Instead of using the likelihood ratio to find the confidence region, this method uses the probability distribution of the true parameter s to find the confidence region with appropriate coverage. It uses an ordering method based on the ratio

$$\begin{aligned} R &= \frac{P(N_{on}|s)}{P(N_{on}|s_{\text{best}})} \\ P(N_{on}|s) &= \frac{(s+b)^{N_{on}} e^{-(s+b)^{N_{on}}}}{N_{on}!} \end{aligned}$$

This paper makes a lookup table for all possible s and N_{on} values, filling in $P(N_{on}|s)$ and $P(N_{on}|s_{\text{best}})$ (found by maximizing P by varying s). With the table completed for many s, the procedure then ranks the table by R.

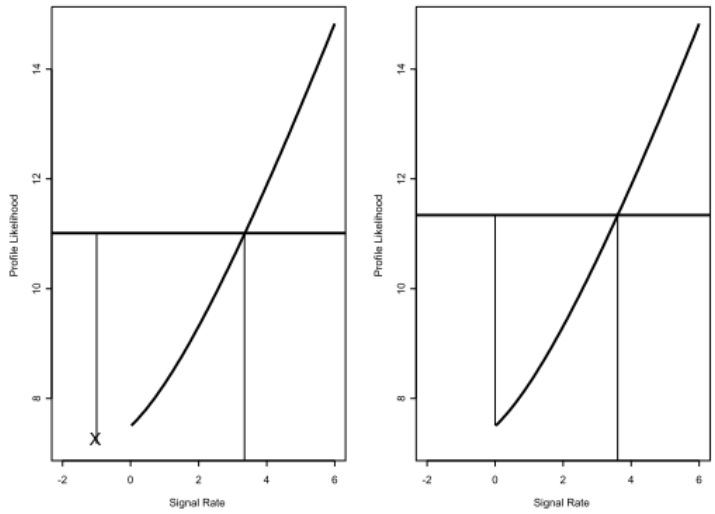

Figure 2: The case $x = 2$, $y = 15$ and $\tau = 5.0$. In the left panel we use the unbounded likelihood method and find a 95% upper limit of 3.35. In the right panel using the bounded likelihood method the 95% upper limit is 3.6.

Figure 6.1:

The confidence limit is found by summing all $P(N_{on}|s)$ in order of R until the confidence limit α (e.g. 90%) is reached for the measured b and N_{on}. The largest and smallest s for given N_{on} are the confidence intervals in the table.

Helene

The basic assumption of Helene is to pick a probability distribution for the signal counts $P(s)$, then solve directly for the upper limit s.

$$\alpha = \int_s^\infty P(s)ds$$

$$P(s) = \begin{cases} N_1 \frac{(s+b)^{Non} e^{-(s+b)^{Non}}}{N_{on}!} & \text{Poisson s, known b} \\ N_2 \frac{e^{-(s-\hat{s})^2/2Non}}{\sqrt{2\pi N_{on}}} & \text{Gaussian s, known b} \\ N_3 \frac{e^{-(s-\hat{s})^2/2\sigma}}{\sqrt{2\pi}\sigma} & \text{Poisson s, Gaussian b} \end{cases}$$

A key nuance: s is never 0, since the first integral is then 1 (note the normalizations N_1, N_2, N_3, set such that $\int_0^\infty P(s)ds = 1$.)

For the gaussian approximation,

$$I(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-x^2/2} dx$$

$$\alpha = \frac{I(\frac{s-\hat{s}}{\sigma})}{I(\frac{-\hat{s}}{\sigma})}$$

Not done here, but s is analytically solvable for all three cases.

6.6 Effective Area

6.7 Energy Threshold

Standard techniques to calculate energy threshold.

1. Function of how many events you throw. When EA gets to 20% statistical uncertainty. Statistical uncertainty is sqrt(counts) that pass cuts in that energy. Method: `CalculatePrecisionEnergyRange`. This is the default selection.

2. Find peak of EA curve - go to 10% of that. what low energy do you get above that?

3. Energy bias has to be below 10% - this energy value is threshold. $Bias = (E_{rec} - E_{true})/E_{true}$

4. Peak of differential counting rate: Take Crab spectrum and multiply by EA. Peak of product is at threshold. Peak is found by taking histogram of

efficiency curve and finding maximum to a gaussian fit over that histogram. This is method used in UL calculation. Option `UL_EnergyThresholdMethod` selects either max bin or the max from the fitted gaussian. Method: `CalculateStandardEnergyRange`.